Kneser's property for a semilinear parabolic partial differential equation with Dirichlet boundary condition (Functional Equations in Mathematical Models)

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Kneser's property for a semilinear parabolic partial differential equation with Dirichlet boundary condition

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1. Introduction. We consider an initial and boundary value problem

$$(\mathrm{E}_n) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \triangle u + F(t,x,u) & \text{for } 0 < t \le T,\ x \in D, u \in \mathbf{R} \\ u(0,x) = u_0(x) & \text{for } x \in \overline{D}, \\ u(t,x) = 0 & \text{for } 0 < t \le T,\ x \in \partial D, \end{cases}$$

where $T > 0$ is a given constant, $D = (0,1)^n \subset \mathbf{R}^n$, $F : [0,T] \times \overline{D} \times \mathbf{R} \to \mathbf{R}$ is continuous and $u_0 \in C(\overline{D}, \mathbf{R})$ satisfies $u_0(x) = 0$ on ∂D. A continuous function $u(t,x)$ defined on $[0,\tau] \times \overline{D}$ will be called a *(mild)* solution of (E_n) when u is expressed by

$$u(t,x) = \int_D G(t,x,y)u_0(y)\,dy + \int_0^t ds \int_D G(t-s,x,y)F(s,y,u(s,y))\,dy,$$

where G is the *fundamental solution* of $\partial u/\partial t = \triangle u$ with $u = 0$ on ∂D.

We shall discuss the Kneser's property for solutions of (E_n). In [2] and [3], we proved that solutions of (E_n) have Kneser's property, where the boundary condition is replaced with Neumann boundary condition and D is assumed to be a bounded domain with smooth boundary.

In this article, we always assume the following assumption (A) to the function F.

(A) $F(t,x,y)$ is expressed by

$$F(t,x,u) = f(t,x,u) + g(t,x,u),$$

where f and g are continuous functions on $[0,T] \times \overline{D} \times \mathbf{R}$ and satisfy

$$(1) \qquad \begin{cases} f(t,x,u) = 0 & \text{for } 0 \le t \le T,\ x \in \partial D, u \in \mathbf{R}, \\ g(t,x,-u) = -g(t,x,u) & \text{for } 0 \le t \le T,\ x \in \overline{D}, u \in \mathbf{R}. \end{cases}$$

Only for simplicity of notations, we shall state our results in the case where $n = 1$, and hence, (E_n) will be reduced to the problem

$$
(\mathrm{E}_1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + F(t,x,u) & \text{for } 0 < t \le T,\ x \in \overline{D} = [0,1], u \in \mathbf{R}, \\ u(0,x) = u_0(x) & \text{for } x \in \overline{D} = [0,1], \\ u(t,0) = u(t,1) = 0 & \text{for } 0 < t \le T, \end{cases}
$$

where u_0 is a continuous function satisfying $u_0(0) = u_0(1) = 0$. The following example shows that solutions of (E_1) are not always unique.

Example. Consider the following problem for $t > 0, x \in [0,1]$ and $u \in \mathbf{R}$.

$$
(\mathrm{E}) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \sqrt{\dfrac{x^4 - 2x^3 + x}{12}}\sqrt{|u|} + \dfrac{12u}{1 + x - x^2}, \\ u(0,x) = 0, \\ u(t,0) = u(t,1) = 0. \end{cases}
$$

It is clear that (E) admits the zero solution $u(t,x) \equiv 0$. Furthermore, it is not difficult to see that

$$
u(t,x) = \frac{t^2(x^2 - x)(x^2 - x - 1)}{48} = \frac{t^2}{4} \cdot \frac{x^4 - 2x^3 + x}{12}
$$

is also a solution of (E).

Remark. The function F in (E) satisfies assumption (A).

2. Compactness of solutions. It is well known (e.g. [1]) that the fundamental solution G for $\partial u/\partial t = \partial^2 u/\partial x^2$ with $u(t,0) = u(t,1) = 0$ is expressed by

$$
(2) \qquad G(t,x,y) = \sum_{k=-\infty}^{k=\infty} \{E(t, x - y + 2k) - E(t, x + y + 2k)\},
$$

where $E(t,\xi) = (4\pi t)^{-1/2} \exp(-\xi^2/4t)$ for $t > 0, \xi \in \mathbf{R}$.

Let X be any metric space. We shall denote by $BC(X,\mathbf{R})$ the Banach space of all bounded and continuous functions on X with the norm $\|\cdot\|$ defined by

$$
(3) \qquad \|v\| = \sup\{|v(x)|; x \in X\}
$$

for $v \in BC(X,\mathbf{R})$. Similarly, for any compact metric space X, we shall denote by

$C(X, \mathbf{R})$ the Banach space of all continuous functions on X with the norm $\|\cdot\|$ given by (3).

By assumption (A), the functions f and g admit a continuous and nondecreasing function $\varphi : [0, \infty) \to (0, \infty)$ with the property that

$$(4) \qquad |f(t, x, u)| \leq \varphi(|u|), \quad |g(t, x, u)| \leq \varphi(|u|)$$

for $(t, x, u) \in [0, T] \times [0, 1] \times \mathbf{R}$.

Now we shall define several extensions of the functions $u_0(x)$, $u(t, x)$, $f(t, x, u)$ and $g(t, x, u)$ in the following way. For a function $u_0 \in C([0, 1], \mathbf{R})$ with $u_0(0) = u_0(1) = 0$, we can easily construct a continuous extension $\hat{u}_0 : \mathbf{R} \to \mathbf{R}$ of u which satisfies that $\hat{u}_0(x)$ is an odd mapping and is 2-periodic. Similarly, for $\tau \in (0, T]$ and for a function $u = u(t, x) \in C([0, \tau] \times [0, 1], \mathbf{R})$ satisfying $u(t, 0) = u(t, 1) = 0$ on $[0, \tau]$, let $\hat{u} = \hat{u}(t, x) \in C([0, \tau] \times \mathbf{R}, \mathbf{R})$ be a continuous extension of u which is an odd mapping and 2-periodic in x for each $t \in [0, \tau]$, while let $\tilde{u} = \tilde{u}(t, x) \in C([0, \tau] \times \mathbf{R}, \mathbf{R})$ be a continuous extension of u which is an even mapping and 2-periodic in x for each $t \in [0, \tau]$. Finally, for the functions f and g satisfying (1), let $\hat{f} = \hat{f}(t, x, u) \in C([0, T] \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$ be an extension of f which is an odd mapping and 2-periodic in x for each $(t, u) \in [0, T] \times \mathbf{R}$, while $\tilde{g} = \tilde{g}(t, x, u) \in C([0, T] \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$ be an extension of g which is an even mapping and 2-periodic in x for each $(t, u) \in [0, T] \times \mathbf{R}$. Here, notice that $\tilde{g}(t, x, u)$ is an odd mapping in u because of (1).

Lemma 1. For a function $u_0 \in C([0, 1], \mathbf{R})$ with $u_0(0) = u_0(1) = 0$, we have

$$\int_D G(t, x, y) u_0(y)\, dy = \int_{\mathbf{R}} E(t, x - y) \hat{u}_0(y)\, dy.$$

Proof. It follows from (2) that

$$\begin{aligned}
&\int_D G(t, x, y) u_0(y)\, dy \\
&= \sum_{k=-\infty}^{k=\infty} \left\{ \int_0^1 E(t, x - y + 2k) u_0(y)\, dy - \int_0^1 E(t, x + y + 2k) u_0(y) \right\} dy \\
&= \sum_{k=-\infty}^{k=\infty} \left\{ \int_{-2k}^{1-2k} E(t, x - z) u_0(z + 2k)\, dz + \int_{-2k}^{-1-2k} E(t, x - z) u_0(-z - 2k) \right\} dz \\
&= \sum_{k=-\infty}^{k=\infty} \left\{ \int_{-2k}^{1-2k} E(t, x - z) \hat{u}_0(z)\, dz + \int_{-1-2k}^{-2k} E(t, x - z) \hat{u}_0(z) \right\} dz
\end{aligned}$$

$$= \int_{\mathbf{R}} E(t, x-y)\hat{u}_0(y)\, dy. \qquad \square$$

Lemma 2. Suppose that (A) holds and that $\tau \in (0, T]$. Then for a function $u \in C([0,\tau] \times [0,1], \mathbf{R})$ satisfying $u(t,0) = u(t,1) = 0$ for $t \in [0,\tau]$, it follows, for $0 \le s \le t \le \tau$, that

$$\int_D G(t-s, x, y) f(s, y, u(s,y))\, dy = \int_{\mathbf{R}} E(t-s, x-y)\hat{f}(s, y, \tilde{u}(s,y))\, dy$$

and

$$\int_D G(t-s, x, y) g(s, y, u(s,y))\, dy = \int_{\mathbf{R}} E(t-s, x-y)\tilde{g}(s, y, \hat{u}(s,y))\, dy.$$

Proof. It is easy to observe that the following equalities hold for each $(s,y) \in [0,\tau] \times \mathbf{R}$.

$$\hat{f}(s, -y, \tilde{u}(s,-y)) = -\hat{f}(s, y, \tilde{u}(s,y)), \quad \hat{f}(s, y+2, \tilde{u}(s, y+2)) = \hat{f}(s, y, \tilde{u}(s,y)),$$

$$\tilde{g}(s, -y, \hat{u}(s,-y)) = -\tilde{g}(s, y, \hat{u}(s,y)), \quad \tilde{g}(s, y+2, \hat{u}(s, y+2)) = \tilde{g}(s, y, \hat{u}(s,y)).$$

By using the similar arguments as in the proof of Lemma 1, we can easily prove the assertion of the lemma. $\square$

Let $h : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying

$$|h(t,x,u)| \le \varphi(|u|) \quad \text{for} \quad (t,x,u) \in [0,T] \times \mathbf{R} \times \mathbf{R}, \tag{5}$$

where $\varphi : [0,\infty) \to (0,\infty)$ is a continuous and nondecreasing function introduced in the above. For this function h, $\tau \in (0,T]$ and for $u \in BC([0,\tau] \times \mathbf{R}, \mathbf{R})$, define a function $H(h,u,\tau)$ on $[0,\tau] \times \mathbf{R}$ by

$$[H(h,u,\tau)](t,x) = \int_0^t ds \int_{\mathbf{R}} E(t-s, x-y) h(s, y, u(s,y))\, dy.$$

By using similar arguments as in the proof of Lemma 1.5 in [2], we can prove the following lemma.

Lemma 3. For any $\tau \in (0,T]$, $u \in BC([0,\tau] \times \mathbf{R}, \mathbf{R})$ and for any function h satisfying (5), we have

$$\begin{aligned} &|[H(h,u,\tau)](t,x) - [H(h,u,\tau)](t',x')| \\ &\qquad \le 8M\sqrt{t}\sqrt{t'-t} + M(t'-t) + 2\sqrt{2}M\sqrt{t}\,|x-x'| \end{aligned}$$

for any $0 \le t < t' \le \tau$ and $x, x' \in \mathbf{R}$, where $M = \sup\{|h(t,x,u(t,x))|; t \in [0,\tau], x \in \mathbf{R}\} \le \varphi(\|u\|) < \infty$.

Theorem 1 (Existence). Suppose that (A) holds. Then for any function $u_0 \in C([0,1], \mathbf{R})$ with $u_0(0) = u_0(1) = 0$, there exists at least one solution $u(t,x)$ of (E_1) on $[0,\tau] \times [0,1]$ for some $\tau > 0$.

Proof. Put $\|u_0\| = M_0$ and take a number L satisfying $L > M_0$. Then we can choose a number $\tau > 0$ so that an inequality

$$M_0 + 2\varphi(L)\tau \le L$$

holds. We denote by V the set of all functions $u \in C([0,\tau] \times [0,1], \mathbf{R})$ which satisfy that $\|u\| \le L$, $u(t,0) = u(t,1) = 0$ and that $u(0,x) = u_0(x)$ for $x \in [0,1]$. Then V is a closed and convex subset of $C([0,\tau] \times [0,1], \mathbf{R})$. For every $v \in V$, we define a mapping $\Psi v : [0,\tau] \times [0,1] \to \mathbf{R}$ by $[\Psi v](0,x) = u_0(x)$ for $x \in [0,1]$ and

$$[\Psi v](t,x) = \int_D G(t,x,y) u_0(y)\,dy + \int_0^t ds \int_D G(t-s,x,y) F(s,y,v(s,y))\,dy$$

for $0 < t \le \tau$, $x \in [0,1]$. Then Ψv belongs to $C([0,\tau] \times [0,1], \mathbf{R})$ and $[\Psi v](t,0) = [\Psi v](t,1) = 0$ for $t \in (0,\tau]$. It folows from Lemmas 1 and 2 that

$$\begin{aligned}[\Psi v](t,x) = & \int_{\mathbf{R}} E(t,x-y)\hat{u}_0(y)\,dy \\ & + \int_0^t ds \int_{\mathbf{R}} E(t-s,x-y)\hat{f}(s,y,\tilde{v}(s,y))\,dy \\ & + \int_0^t ds \int_{\mathbf{R}} E(t-s,x-y)\tilde{g}(s,y,\hat{v}(s,y))\,dy,\end{aligned} \tag{6}$$

thus we have

$$\begin{aligned}|[\Psi v](t,x)| \le & M_0 + \int_0^t ds \int_{\mathbf{R}} E(t-s,x-y)\varphi(\|\tilde{v}\|)\,dy \\ & + \int_0^t ds \int_{\mathbf{R}} E(t-s,x-y)\varphi(\|\hat{v}\|)\,dy \\ \le & M_0 + 2\varphi(L)\tau \le L\end{aligned}$$

because $\int_{\mathbf{R}} E(t,x-y)\,dy = 1$. Therefore, we obtain that $\Psi(V) \subset V$. It follows from (6) and Lemma 3 that $\Psi(V)$ is relatively compact, and hence, we can find a fixed point u in V by Shauder's fixed point theorem. Clearly, u is a solution of (E_1), which completes the proof. $\square$

Lemma 4. Suppose that (A) holds. Then there exist two numbers $\tau > 0$ and $M > 0$ such that every solution u of (E_1) exists and satisfies $|u(t,x)| \le M$ on $[0,\tau] \times [0,1]$.

Proof. Put $\|u_0\| = M_0$. Then any solution u of (E_1) satisfies

$$\begin{aligned} |u(t,x)| &\le M_0 + 2\int_0^t ds \int_{\mathbf{R}} E(t-s, x-y)\varphi(\|u(s)\|)\,dy \\ &\le M_0 + 2\int_0^t \varphi(\|u(s)\|)\,ds \end{aligned}$$

for $t > 0$ and $x \in [0,1]$ as long as u exists, where $\|u(s)\| = \sup\{|u(s,y)|; y \in [0,1]\}$. Therefore, it follows that

$$\|u(t)\| \le M_0 + 2\int_0^t \varphi(\|u(s)\|)\,ds.$$

If we put $v(t) := \|u(t)\|$ and $w(t) := M_0 + 2\int_0^t \varphi(v(s))\,ds$ for $t > 0$, then we have $v(t) \le w(t)$ and $w'(t) = 2\varphi(v(t)) \le 2\varphi(w(t))$. By the comparison theorem in the theory of ordinary differential equations, the maximal solution $p(t)$ of $p' = 2\varphi(p)$ with $p(0) = M_0$ exists on $[0,\tau]$ for some $\tau > 0$ and an inequality $p(\tau) \ge p(t) \ge w(t)$ holds on $[0,\tau]$. By putting $M = p(\tau)$, we have the assertion. □

3. Kneser's property. For the functions f and g satisfying (1) and for $m \in \mathbf{N}$, we put

$$f_m(t,x,u) = \frac{m}{2}\int_{u-\frac{1}{m}}^{u+\frac{1}{m}} f(t,x,v)\,dv, \quad g_m(t,x,u) = \frac{m}{2}\int_{u-\frac{1}{m}}^{u+\frac{1}{m}} g(t,x,v)\,dv.$$

Then $f_m(t,x,u) = 0$ for $x = 0,1$, while $g_m(t,x,-u) = -g_m(t,x,u)$ by virtue of (1). It is easy to see that $\{f_m\}$ and $\{g_m\}$ converge, respectively, to f and g uniformly on every compact set in $[0,T] \times [0,1] \times \mathbf{R}$. Clearly, f_m and g_m are locally Lipschitz continuous in u. Moreover, by the mean value theorem in integration, we have

$$\begin{aligned} |f_m(t,x,u)| &\le \frac{m}{2}\int_{u-\frac{1}{m}}^{u+\frac{1}{m}} |f(t,x,v)|dv \le \frac{m}{2}\int_{u-\frac{1}{m}}^{u+\frac{1}{m}} \varphi(|v|)\,dv \\ &= \varphi\left(|u+\theta/m|\right) \le \varphi(|u|+1), \end{aligned}$$

where θ is a suitable number satisfying $-1 < \theta < 1$. By replacing $\varphi(s+1)$ by $\varphi(s)$, we may assume that $|f_m(t,x,u)| \le \varphi(|u|)$. Similarly, we may also assume that $|g_m(t,x,u)| \le \varphi(|u|)$.

Theorem 2. Suppose that (A) holds and that $u_0 \in C([0,1], \mathbf{R})$ is an arbitrary function satisfying $u_0(0) = u_0(1) = 0$. Then a family

$$\mathcal{F} = \{u \in C([0,\tau] \times [0,1], \mathbf{R});\ u \text{ is a solution of } (\mathrm{E}_1)\}$$

is compact and connected in $C([0,\tau] \times [0,1], \mathbf{R})$ when $\tau > 0$ is sufficiently small.

Proof. By Lemma 4, there exist $\tau > 0$ and $M > 0$ such that every solution u of (E_1) exists and satisfies $|u(t,x)| \leq M$ on $[0,\tau] \times [0,1]$. For this $\tau > 0$, we shall prove the assertion of the theorem.

It suffices to show that $\mathcal{F}$ is connected because the compactness of $\mathcal{F}$ is obvious by Lemma 3. Suppose that $\mathcal{F}$ is not connected. Then there exist an open set $\mathcal{O}$ and two nonempty compact sets $\mathcal{F}_1$ and $\mathcal{F}_2$ in $C([0,\tau] \times [0,1], \mathbf{R})$ such that

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}, \quad \mathcal{F}_1 \subset \mathcal{O}, \quad \mathcal{F}_2 \cap \overline{\mathcal{O}} = \emptyset.$$

Let u_1 and u_2 be any elements in $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Then, for each $m \in \mathbf{N}$, u_i is a solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + H_i(t,x,u), \quad (i = 1,2),$$

where

$$H_i(t,x,u) = F(t,x,u_i(t,x)) - F_m(t,x,u_i(t,x)) + F_m(t,x,u)$$

and

$$F_m(t,x,u) = f_m(t,x,u) + g_m(t,x,u).$$

Let m be fixed. For any $\theta \in [0,1]$, define $\Phi_\theta(t,x,u)$ by

$$\Phi_\theta(t,x,u) = (1-\theta)H_1(t,x,u) + \theta H_2(t,x,u).$$

Then $\Phi_\theta(t,x,u)$ is expressed by

$$\Phi_\theta(t,x,u) = G_m(t,x) + f_m(t,x,u) + g_m(t,x,u),$$

where

$$G_m(t,x) = (1-\theta)\{F(t,x,u_1(t,x)) - F_m(t,x,u_1(t,x))\}$$
$$+ \theta\{F(t,x,u_2(t,x)) - F_m(t,x,u_2(t,x))\}.$$

Here, we notice that $G_m(t,0) = G_m(t,1) = 0$. Since $\{G_m(t,x)\}$ converges to 0 uniformly on $[0,\tau] \times [0,1]$ as $m \to \infty$, we may assume that $|G_m(t,x)| \leq 1$ for $m \in \mathbf{N}$

by taking a subsequence if necessary. Therefore, we may also assume that

$$|G_m(t,x) + f_m(t,x,u)| \le \varphi(|u|)$$

by replacing $1 + \varphi(s)$ by $\varphi(s)$.

For any fixed $m \in \mathbf{N}$, a problem

$$(\mathrm{E}_\theta) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \Phi_\theta(t,x,u) & \text{for } 0 < t \le \tau,\ x \in [0,1], u \in \mathbf{R}, \\ u(0,x) = u_0(x) & \text{for } x \in [0,1], \\ u(t,0) = u(t,1) = 0 & \text{for } 0 < t \le \tau \end{cases}$$

has a unique solution $v_\theta(t,x)$ because $\Phi_\theta(t,x,u)$ is locally Lipschitz continuous in u. Evidently, $v_0 = u_1$ and $v_1 = u_2$. Moreover, it is not difficult to verify that a mapping $\theta \mapsto v_\theta$ is continiuos from $[0,1]$ into $C([0,\tau] \times [0,1], \mathbf{R})$, and hence, there exists a $\theta \in [0,1]$ such that $v_\theta \in \partial\mathcal{O}$. We denote these θ and v_θ by θ_m and u_m, respectively. Then u_m is a solution of (E_{θ_m}) and a relation $u_m \in \partial\mathcal{O}$ holds. It follows from Lemma 3 that $\{u_m\}$ is equicontinuous on $[0,\tau] \times [0,1]$, and hence, we may assume that $\{u_m\}$ converges uniformly to some $u \in C([0,\tau] \times [0,1], \mathbf{R})$ by taking a subsequence if necessary. Since $\{\Phi_{\theta_m}\}$ converges to $f + g$ uniformly on every compact set in $[0,\tau] \times [0,1] \times \mathbf{R}$, u is a solution of (E_1), which implies that $u \in \partial\mathcal{O}$ and $u \in \mathcal{F}$. This is a contradiction. $\square$

The following corollary is a direct consequence of Theorem 2.

Corollary. Under the same assumptions as in Theorem 2, a set

$$\mathbf{F} = \{u(\tau) \in C([0,1], \mathbf{R});\ u \text{ is a solution of } (\mathrm{E}_1)\}$$

is compact and connected in $C([0,1], \mathbf{R})$ when $\tau > 0$ is sufficiently small.

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